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Double Injection in Insulators. II. Further Analytic Results with Negative Resistance

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The problem of double injection under varying lifetime conditions is analytically intractable when the physically important space charge is included in the analysis. The regional-approximation method is used to obtain approximate analytic solutions for two prototype insulator problems, both involving a single set of recombination centers and exhibiting a current-controlled negative resistance. In the first problem the centers are partially filled in thermal equilibrium; in the second problem they are completely filled. For both problems we assume that the capture cross section of a filled center for a free hole greatly exceeds that of an empty center for a free electron. This leads to a hole lifetime which increases with injection level, this being the source of the negative resistance. Earlier work based on the assumption of local neutrality is simplified in its domain of validity. Also, a previous analytic result for the first problem, yielding the threshold voltage at zero current, is derived in a simple manner which also yields the complete field and density distributions, previously unavailable.

I. INTRODUCTION

DOUBLE injection under varying lifetime conditions is of unusual interest because of an associated negative resistance. Unfortunately, the theoretical problem, stated in its full glory, is hopelessly intractable. Meaningful progress in understanding double-injection phenomena is possible only with the help of simplifying assumptions and judicious approximations. A particularly simplified model of double injection exhibiting a current-controlled negative resistance was earlier studied by one of the authors (M.A.L.) in a paper[1] hereafter referred to as I. This model incorporated a single set of recombination centers, initially filled, and for which $\bar{\sigma}_p \gg \bar{\sigma}_n$, these being the average capture cross sections for holes by filled centers and for electrons by empty centers, respectively. The key assumption of local neutrality was made to achieve analytic tractability and an outstanding feature of the solution thereby generated was a threshold voltage for current flow. As discussed in I, the analytic solution obtained is consistent with the neutrality assumption at high current levels but is necessarily inconsistent at threshold (low) currents. A similar analysis was made, by Keating,[2] of the more general situation in which the single set of recombination centers is only partially filled at the outset, and again assuming local neutrality to achieve analytic tractability. In this case a completely unphysical result is predicted at low currents, namely, an infinite threshold voltage for current flow. However, again, at sufficiently high currents the analytic solution is consistent with the neutrality assumption and is physically correct. This same model was investigated by Ashley[3] in his PhD thesis, including space charge in the defining equations. The consequent loss of analytic tractability forced the use of digital computation to obtain the current-voltage (I–V) characteristic for a few specific cases. Ashley did succeed in obtaining one analytic result for this model, namely, the threshold voltage for current flow, i.e., the voltage in the limit of zero current.

In this report we study analytically, with the inclusion of space charge, both of the above-mentioned models: a single set of recombination centers partially filled initially[2,3] and completely filled initially.[1] Clearly, to progress beyond the previous understanding of these models we must inject a new approximation procedure into the theory. A procedure which works exceptionally well with the generally intractable problems of injection currents in solids is the regional approximation method. This method is based on the simple observation that where there are several terms in an equation "competing" with each other, each being a function of position, it is generally possible to divide up the volume of the solid into separate regions, in each of which a single term dominates. Within each such region the basic approximation is made of neglecting all terms within the competing group except the one term which is dominant in that region. It turns out, generally, that inside each region the simplified set of equations has an analytic solution. It remains then only to connect the solutions in adjacent regions, and this is done via a continuity requirement, say on the electric field intensity.

The regional approximation method is an attractive procedure not only because it brings the intractable into the realm of the tractable, which would itself be sufficient justification for its use, but in so doing it leans heavily on physical considerations and necessarily exposes the underlying physics governing the behavior in each region. All of the interesting new results of this paper are obtained by its use. These new results include

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[1] M. A. Lampert, Phys. Rev. **125**, 126 (1962).

[2] P. N. Keating, Phys. Rev. **135**, A1407 (1964).

[3] K. L. Ashley, Ph.D. thesis, Carnegie Institute of Technology, 1963 (unpublished).

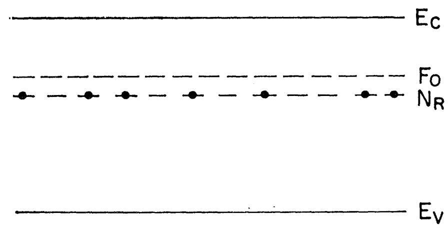

FIG. 1. Schematic energy-band diagram for the problem of double injection into an insulator with a single set of recombination centers partially filled in thermal equilibrium.

a complete, analytic specification, for both models, of the I–V characteristic over the entire range of currents, from zero current up to "high" currents, i.e., currents at which the semiconductor, constant-lifetime, injected-plasma description[4] prevails. A detailed picture of the electric field and density distributions is available at any current level. In particular, the transition from space-charge-dominated to neutrality-dominated current flow is graphically traced. Ashley's threshold-voltage result[3] for the partially filled center model is reproduced, in the appropriate limit, by a very much simpler analysis which, at the same time, yields a simple picture of the field and density distributions (not available from Ashley's analysis). Finally, the unwieldy algebraic results of I for the completely filled center model, valid in the neutrality-dominant realm of currents, are replaced by a considerably simpler set of algebraic results which are, for practical purposes, identical to the earlier results.

II. RECOMBINATION CENTERS PARTIALLY FILLED, INITIALLY

This problem is illustrated schematically by the energy-band diagram of Fig. 1. There is a single set of recombination centers lying not too far from the Fermi level, and therefore partially occupied by electrons in thermal equilibrium. The average free-carrier capture cross sections $\bar{\sigma}_p$ and $\bar{\sigma}_n$ for holes and electrons, respectively, are such that $\bar{\sigma}_p \gg \bar{\sigma}_n$. This inequality would be the normal expectation if, for example, the centers are acceptorlike, that is, negatively charged when electron-occupied. In that case we might imagine that the negative charge held in the centers in thermal equilibrium is precisely cancelled by the positive ionic charge of compensating donors, the donors being so shallow as to play no further role in the electronic behavior of the insulator (and therefore not shown in Fig. 1).

The equations governing this problem, written in mks units, are the current-flow equation

$$J = e\mu_n n\mathcal{E} + e\mu_p p\mathcal{E} = \text{const}, \tag{1}$$

the Poisson equation

$$(\epsilon/e)(d\mathcal{E}/dx) = p + (p_R - p_{R,0}) - n, \tag{2}$$

[4] M. A. Lampert and A. Rose, Phys. Rev. 121, 26 (1961).

the particle-conservation equations

$$\mu_n(d/dx)(n\mathcal{E}) = -r = -\mu_p(d/dx)(p\mathcal{E}), \tag{3}$$

and the recombination-kinetic expressions

$$\begin{aligned} r = n/\tau_n = p/\tau_p, \quad 1/\tau_n = \langle v\sigma_n\rangle p_R, \\ 1/\tau_p = \langle v\sigma_p\rangle n_R, \quad p_R + n_R = N_R. \end{aligned} \tag{4}$$

The two equations in (3) are readily combined to give

$$(d/dx)[(n-p)\mathcal{E}] = (a+1)n/\mu_p\tau_n = (a+1)p/\mu_p\tau_p, \quad a = \mu_p/\mu_n. \tag{5}$$

The solution to these equations is subject to the boundary condition

$$\mathcal{E} = 0 \quad \text{at } x=0 \quad \text{and at } x=L. \tag{6}$$

In the above, J is the total current density, e is the magnitude of electronic charge, n and p are the free-electron and free-hole densities respectively, μ_n and μ_p are their respective drift mobilities, $\mathcal{E}$ is the electric field intensity, ϵ is the static dielectric constant, p_R and $p_{R,0}$ are the densities of empty recombination centers under injection conditions and in thermal equilibrium, respectively, r is the recombination-rate density, τ_n and τ_p are the free-electron and free-hole lifetimes, respectively, v is the microscopic free-carrier kinetic velocity, $\sigma_n = \sigma_n(v)$ and $\sigma_p = \sigma_p(v)$ are the (velocity-dependent) electron- and hole-capture cross sections, respectively, $\langle\ \rangle$ signifies that an average is to be taken over the free-carrier velocity distribution, and N_R is the density of recombination centers. The anode, which is the hole-injecting contact, is located at $x=0$ and the cathode, which is the electron-injecting contact, at $x=L$.

In writing Eqs. (1)–(6) the following assumptions have been made:

(i) The current is volume-controlled, i.e., the contacts impose no significant constraints on the currents entering or leaving the insulator.

(ii) Diffusion currents are negligible.

These two assumptions are linked together and are both violated in the immediate neighborhood of an injecting contact. (For, a contact which is injecting for electrons will block the egress of holes, and vice versa, particularly under one-dimensional planar-flow conditions.) Therefore, the solution to Eqs. (1)–(6) gives a good description of the double-injection current flow only if the separation L between injecting contacts is sufficiently large. Since the buildup of plasma near an injecting contact leads to a diffusion-dominated current-flow there, "sufficiently large" means large compared with an ambipolar diffusion length L_a. Within the negative-resistance regime of the I–V characteristic, namely, for $J<J_M$ in Figs. 3 and 5, over a significant volume of the insulator the hole lifetime is very short and correspondingly L_a is very small. Therefore assumptions (i) and (ii) are not likely to lead to difficulties in this regime. In the

semiconductor injected-plasma regime, namely, for $J > J_M$ in Figs. 3 and 5, the hole lifetime is everywhere comparable to the electron lifetime and therefore much longer than at lower injection levels. Correspondingly L_a is substantially larger than at lower injection levels. Therefore this is the regime in which the use of assumptions (i) and (ii) is apt to produce the largest errors. Fortunately, the errors incurred in neglecting diffusion currents in the study of the constant-lifetime semiconductor injected-plasma problem are well understood[5,6] and easily taken into account. In particular the appropriateness of assumptions (i) and (ii) when $L/L_a \gg 1$ has been well established, in both theoretical[5,6] and experimental[7] studies. The boundary conditions (6) are the usual ones employed in a simplified theory neglecting diffusion.

(iii) Low-field field-independent mobility conditions obtain. This assumption facilitates the analytic approach. With the possible exception of drift-velocity saturation, virtually any other field dependence of the mobility will nullify the hard-won analytic tractability. It is a reasonable assumption in insulator studies. Caution is in order in the use of this assumption in studies with high-mobility semiconductors, such as InSb and InAs and, at low temperatures, Ge and Si.

(iv) Thermal reemission of carriers from the recombination center is negligible.

More specifically, the rate of thermal reemission of carriers from the recombination centers to a band is assumed to be small compared to their capture rate from that band. If the converse is true over a significant region of the insulator, then this very likely signifies that the corresponding regime in the I–V characteristic is dominated by a one-carrier space-charge-limited current flow. With the use of assumption (iv) one forecloses the possibility of describing such a regime by his analysis and must, perforce, tack on the description in a strictly *ad hoc* manner. An example of this procedure is cited at the end of Sec. II. The failure of assumption (iv) can also have a quite different, but rather more remote, interpretation; namely, the failure may correspond to a situation in which net recombination is a relatively small difference between large capture and thermal reemission rates. An interesting discussion of such situations has been given by Rose[8] in the context of photoconductivity problems. Assumption (iv) has been made by all authors studying the double-injection problem. Abandonment of this assumption would greatly escalate the complications in the problem, possibly beyond the reach of analysis, and certainly beyond the reach of convenience.

(v) The thermally generated free carriers can be neglected.

This assumption is easily satisfied in insulators, but it certainly need not be satisfied in semiconductors. With the latter materials, an obvious manifestation of the presence of thermal free carriers will be the observation of Ohm's law at low current levels. Not obvious, but easily derived,[3,9] is a square-law regime immediately following Ohm's law in the I-V characteristic. Following the square-law regime, for $\bar{\sigma}_p \gg \bar{\sigma}_n$, is a negative-resistance regime for which there is, as yet, no adequate theory.

(vi) One-dimensional planar flow current conditions obtain.

Ordinarily this would rank as a cut-and-dried statement which need not be elevated to the status of an assumption. However, it is known[10] that under *homogeneous* current-flow conditions (which do *not* obtain in the double-injection problem) in a regime of current-controlled negative-resistance planar current flow is unstable against filament formation. In the corresponding experimental situation,[11] the filaments are present under some conditions, absent under others.

With regard to assumptions (v) and (vi), and to a lesser extent (iii), it is clear that the analyses of this paper do not encompass all of the interesting physical phenomena associated with double-injection negative resistance. Unfortunately, the complexity of the full double-injection problem is so great that there is no hope of building up a fund of insight except by proceeding a step at a time.

Since the set of Eqs. (1)–(6) is beyond the reach of exact analytic solution we use the regional approximation method to obtain an approximate analytic solution. In order to map out the strategy for the designation of the separate regions we must consider the gross physical behavior excepted of the model, particularly at low current levels. In the vicinity of the anode, current is carried under conditions of approximate local neutrality. For, the hole capture rate being large near the anode, the electron capture rate must be precisely as large, the two rates being exactly equal everywhere in the insulator in the steady state, because of assumption (iv). This requires large electron densities near the anode, with $n > p$ everywhere because of the assumption that $\bar{\sigma}_p \gg \bar{\sigma}_n$ ($n = p + n_{R,0}$ in region I adjacent to the anode, and $n \gg p$ in region II following region I, as discussed below). Such large electron densities, far removed from the cathode, are most easily achieved under conditions of neutrality, in which case each

[5] See R. Baron [J. Appl. Phys. **39**, 1435 (1968)] for a computer study of these errors.
[6] See R. B. Schilling aud M. A. Lampert (unpublished) for an analytic study of these errors using the regional-approximation method.
[7] O. J. Marsh, J. W. Mayer, and R. Baron, Appl. Phys. Letters **5**, 74 (1965).
[8] A. Rose, *Progress in Semiconductors* (Heywood and Co., Ltd., London, 1957), Vol. 2, p. 109.
[9] K. L. Ashley and A. G. Milnes, J. Appl. Phys. **35**, 369 (1964).
[10] B. K. Ridley, Proc. Phys. Soc. (London) **82**, 954 (1963).
[11] A. M. Barnett and A. G. Milnes, J. Appl. Phys. **37**, 4215 (1966).

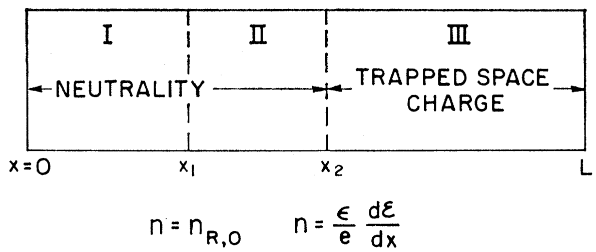

FIG. 2. Schematic regional-approximation diagram for the problem of double injection into an insulator with a single set of recombination centers partially filled in thermal equilibrium.

trapped hole is compensated by a free electron deriving from injection at the cathode. (Effectively, it is *as if* the electron in the center were transferred to the conduction band as a result of hole capture, although this mode of description is, of course, only a manner of speaking.) Immediately adjacent to the anode this region of neutrality has the further property, again because $\bar{\sigma}_p \gg \bar{\sigma}_n$, that the recombination centers have been largely depopulated by hole capture: $p_R \simeq N_R$ and, correspondingly, $n > n_{R,0} = N_R - p_{R,0}$. We call this region I. Region I ends at $x = x_1$, where $n(x) = n_{R,0}$. Adjacent to this region is region II, over which local neutrality still obtains but the electron transfer from the recombination centers to the conduction band is incomplete: $n_{R,0} > n \gg p$, the latter inequality again following from $\bar{\sigma}_p \gg \bar{\sigma}_n$. Region II ends at $x = x_2$ where the assumption of local neutrality runs out of self-consistency, that is, where the neglected space charge becomes comparable to the terms retained in the Poisson equation $(\epsilon/e)(d\mathcal{E}/dx)_2 = n(x_2)$. Region III, encompassing the remaining volume of the insulator, is a region dominated by space charge. Further, since both empty and filled recombination centers are available in this region to trap injected electrons and holes, respectively, the space charge in region III is held largely in the recombination centers. The three regions of the problems are illustrated schematically in Fig. 2.

We proceed to a detailed consideration of the three regions:

Region I. $(0 \leq x \leq x_1)$: $n > n_{R,0}$, $p_R \simeq N_R$.

In place of the Poisson equation (2) we have local neutrality, which for $p_R = N_R$, is specified by

$$n - p = N_R - p_{R,0} = n_{R,0}. \tag{7}$$

Using (7), (1) can be rewritten as

$$J = e\mu_n \mathcal{E}(a+1)n - e\mu_p n_{R,0}\mathcal{E}, \tag{8}$$

and (5) can be rewritten as

$$n_{R,0}\frac{d\mathcal{E}}{dx} = \frac{(a+1)n}{\mu_p \tau_h}, \quad 1/\tau_h = \langle v\sigma_n \rangle N_R, \tag{9}$$

where τ_h is the common lifetime for electrons and holes at high plasma density levels, namely, $n \approx p > n_{R,0}$.

The combination of (8) and (9) yields the characteristic differential equation for this region

$$\frac{\mathcal{E}}{\mathcal{E}+S_0}d\mathcal{E} = T dx, \quad S_0 = \frac{J}{e\mu_p n_{R,0}}, \quad T = \frac{1}{\mu_n \tau_h}, \tag{10}$$

with solution

$$\mathcal{E} - S_0 \ln[(\mathcal{E}+S_0)/S_0] = Tx, \tag{11}$$

satisfying the boundary condition: $\mathcal{E} = 0$ at $x = 0$.

For the potential at position x in this region we have, using (10),

$$V(x) = \int_0^x \mathcal{E} dx = \int_0^{\mathcal{E}} \mathcal{E}\frac{dx}{d\mathcal{E}} d\mathcal{E} = \frac{1}{T}\left\{\tfrac{1}{2}\mathcal{E}^2 - S_0\mathcal{E} + S_0^2 \ln\frac{\mathcal{E}+S_0}{S_0}\right\}. \tag{12}$$

At the right-hand end of region I,

$$x = x_1: \quad n_1 = n(x_1) = n_{R,0}. \tag{13}$$

$$x_1 = (S_0/T)\{a - \ln(1+a)\}, \qquad \mathcal{E}_1 = \mathcal{E}(x_1) = J/e\mu_n n_{R,0} = aS_0. \tag{14}$$

For the voltage drop V_{I} across region I we have

$$V_{\mathrm{I}} = (S_0^2/T)\{\tfrac{1}{2}a^2 - a + \ln(a+1)\}. \tag{15}$$

Region I just fills the insulator when $x_1 = L$. This marks the high-current low-voltage end of the negative resistance regime in the I–V characteristic. The corresponding current and voltage are

$x_1 = L$:

$$J_M = h(a)\frac{en_{R,0}L}{\tau_h}, \quad h(a) = \frac{a}{a - \ln(1+a)}, \qquad V_M = \frac{1}{g(a)}\frac{L^2}{\mu_p \tau_h}, \quad g(a) = \frac{\{a - \ln(1+a)\}^2}{a\{\tfrac{1}{2}a^2 - a + \ln(1+a)\}}. \tag{16}$$

At lower currents we have, from (14)–(16),

$$J \leq J_M: \quad x_1 = L(J/J_M), \quad V_{\mathrm{I}} = V_M(J/J_M)^2. \tag{17}$$

At higher currents (10)–(12), with $x = L$, $\mathcal{E} = \mathcal{E}_c = \mathcal{E}(L)$, and $V = V(L)$, give the following parametric representation of the I–V characteristic:

$$J > J_M: \qquad \mathcal{E}_c - S_0 \ln\frac{\mathcal{E}_c + S_0}{S_0} = TL, \tag{18}$$

$$J > J_M: \quad \frac{1}{T}\left\{\tfrac{1}{2}\mathcal{E}_c^2 - S_0\mathcal{E}_c + S_0^2 \ln\frac{\mathcal{E}_c+S_0}{S_0}\right\} = V. \tag{19}$$

For $J \gg J_M$, $\mathcal{E}_c/S_0 \ll 1$ and the logarithmic term in (18) and (19) can be expanded. Then (18) and (19) simplify, respectively, to $\mathcal{E}_c^2/2S_0 \simeq TL$ and $\mathcal{E}_c^3/3TS_0$

$\simeq V$, which combine to give

$$J \gg J_M: \quad J \simeq (9/8) e n_{R,0} \mu_n \mu_p \tau_h (V^2/L^3), \tag{20}$$

which is the characteristic square law[4] for a plasma injected into a semiconductor. In effect, the transfer of electrons from the filled centers to the conduction band, through hole capture, has electronically transformed the insulator into a semiconductor.

So much for the high-current positive-resistance regime of the I–V characteristic. We return to the negative-resistance regime.

Region II. $(x_1 \leq x \leq x_2)$: $n_{R,0} > n \gg p$, $n_R \simeq n_{R,0}$.

Local neutrality continues to hold in this region, in which case the Poisson equation (2) reduces to

$$n = p_R - p_{R,0} = n_{R,0} - n_R, \tag{21}$$

where we have taken p to be negligible, as dictated by the requirement that electron and hole capture rates be equal, and our assumption that $\bar{\sigma}_p \gg \bar{\sigma}_n$. On this same account (1) simplifies to

$$J = e\mu_n n \mathcal{E}. \tag{22}$$

Multiplying both sides of the recombination-kinetic relation (4), namely, $p\langle v\sigma_p\rangle n_R = n\langle v\sigma_n\rangle p_R$, by $e\mu_p \mathcal{E}$, and substituting from (22) we obtain

$$J_p \approx aJ \frac{\langle v\sigma_n\rangle}{\langle v\sigma_p\rangle} \frac{p_R}{n_R}. \tag{23}$$

Now using

$$(d/dx)(p_R/n_R) = (N_R/n_R^2)(dp_R/dx) = (N_R/n_R^2)(dn/dx)$$

from (21), it follows that

$$\frac{dJ_p}{dx} = \frac{aJ\langle v\sigma_n\rangle N_R}{\langle v\sigma_p\rangle n_R^2} \frac{dn}{dx}. \tag{24}$$

Now, from (3), $dJ_p/dx = -er = -en\langle v\sigma_n\rangle p_R$. Combining this with (24), and using (21), we obtain the characteristic differential equation for this region

$$-\frac{dn}{n(n+p_{R,0})} = \frac{e\langle v\sigma_p\rangle n_R^2}{aJN_R} dx \simeq \frac{e\langle v\sigma_p\rangle n_{R,0}^2}{aJN_R} dx. \tag{25}$$

Note that we have replaced n_R by $n_{R,0}$ in (25). This is a good approximation, referring to Eq. (21), because $n < n_{R,0}$ over all of region II, and $n < \frac{1}{2} n_{R,0}$ over most of region II, namely, everywhere except near $x = x_1$. [In actuality, it is possible to deal with n_R exactly in (25), namely writing $n_R = n_{R,0} - n$ from (21), but the amount of additional algebraic manipulation then required is very large whereas the accompanying gain in accuracy is insignificant.] On the other hand we do not replace $n + p_{R,0} = p_R$ by $p_{R,0}$ on the left-hand side of (25) precisely because we have made no assumption about the relative magnitudes of $p_{R,0}$ and n in region II.

From (22), $dn = -Jd\mathcal{E}/e\mu_n\mathcal{E}^2$, whence (25) may also be written as

$$\frac{d\mathcal{E}}{\mathcal{E} + J/e\mu_n p_{R,0}} = \frac{e\langle v\sigma_p\rangle n_{R,0}^2 p_{R,0}}{aJN_R} dx. \tag{26}$$

The solution to this equation, in dimensionless form, is given below as Eq. (43).

The right-hand terminus x_2 of region II is taken where the neglected space charge overtakes the retained terms in the Poisson equation

$$x = x_2: \quad n_2 = n(x_2) = (\epsilon/e)(d\mathcal{E}/dx)_2. \tag{27}$$

If the local neutrality approximation is enforced throughout the entire insulator, that is, if (27) is ignored then for $J < J_M$ regions I and II fill the insulator, region II extending up to the cathode at $x = L$. In this case, because of the presence of J in the denominator in (25) and (26), the voltage across the insulator becomes infinite as J goes to zero.[2]

Region III. $(x_2 \leq x \leq L)$: $p_R - p_{R,0} \gg n \gg p$.

This region is dominated by *trapped* space charge, the appropriate approximation to the Poisson equation (2) being

$$\frac{\epsilon}{e} \frac{d\mathcal{E}}{dx} \simeq p_R - p_{R,0}. \tag{28}$$

The current equation is (22).

In contrast to region II [see sentence following (23)], here $(d/dx)(p_R/n_R) = (N_R/n_R^2)(dp_R/dx) \simeq (\epsilon N_R/en_R^2) \times (d^2\mathcal{E}/dx^2)$, from (28). Using this result in (23), which is equally valid in region III, we obtain

$$\frac{dJ_p}{dx} \simeq \frac{\epsilon aJ\langle v\sigma_n\rangle N_R}{e\langle v\sigma_p\rangle n_R^2} \frac{d^2\mathcal{E}}{dx^2}. \tag{29}$$

The combination of (3) and (22) gives

$$\frac{dJ_p}{dx} \simeq -\frac{J\langle v\sigma_n\rangle p_R}{\mu_n \mathcal{E}}. \tag{30}$$

Equations (29) and (30) together yield the characteristic differential equation for this region

$$\mathcal{E}\frac{d^2\mathcal{E}}{dx^2} \simeq -\frac{e\langle v\sigma_p\rangle n_R^2 p_R}{\epsilon\mu_p N_R} \simeq -\frac{e\langle v\sigma_p\rangle n_{R,0}^2 p_{R,0}}{\epsilon\mu_p N_R}. \tag{31}$$

The replacement of n_R by $n_{R,0}$ and p_R by $p_{R,0}$ are good approximations, in this region, at finite currents [the former has already been used in Eq. (25)]. Their accuracy becomes greater, the lower the current, and this procedure has *exact* validity at zero current.

At very low currents, region III effectively fills the entire insulator. In this regime we can ignore (10) and (26) and obtain the I–V characteristic from (31) alone. An exact solution to (31), in dimensionless form, is given below, Eq. (48). Here we note only the salient features of the solution. At the outset, the absence of J from (31) already tells us that there is a threshold voltage for current flow. Further, we can estimate this threshold voltage V_{TH} reasonably well by the simplest conceivable approximation, namely, a bare dimensional analysis of (31), replacing $\mathcal{E}d^2\mathcal{E}/dx^2$ by $-V_{\mathrm{TH}}^2/L^4$. This gives

$$V_{\mathrm{TH}} \simeq \left\{ \frac{e\langle v\sigma_p\rangle L^4 n_{R,0}^2 p_{R,0}}{\epsilon\mu_p N_R} \right\}^{1/2}. \tag{32}$$

The exact result, (52), namely, the solution of (31) subject to boundary conditions (6) differs from (32) only through the numerical factor $1/\sqrt{4\pi}$. Note that if a parabolic approximation is used for $\mathcal{E}$, *namely*, $\mathcal{E}=4\mathcal{E}_{\max}x(L-x)/L^2$, then $d^2\mathcal{E}/dx^2=-8\mathcal{E}_{\max}/L^2$ and $\overline{\mathcal{E}}=\frac{2}{3}\mathcal{E}_{\max}$, so that $\overline{\mathcal{E}d^2\mathcal{E}/dx^2}=-12V_{\mathrm{TH}}^2/L^4$, where a bar over a quantity denotes the average of that quantity taken over the insulator. Using this in (31) multiplies the right-hand side of (32) by $1/\sqrt{12}$, which is very close to the correct factor $1/\sqrt{4\pi}$. The parabolic approximation for $\mathcal{E}$ is very close to the correct distribution.

For further discussion of the problem it is convenient to switch to dimensionless variables:

$$x=x^*w\,,\quad \mathcal{E}=\mathcal{E}^*u\,,\quad V=V^*v\,,$$

$$x^*=\frac{aN_RJ}{e\langle v\sigma_p\rangle n_{R,0}^2 p_{R,0}}\,,\quad \mathcal{E}^*=\frac{aN_RJ}{\{e\langle v\sigma_p\rangle n_{R,0}^2 p_{R,0}\epsilon\mu_p N_R\}^{1/2}}\,,$$

$$V^*=\frac{(aN_RJ)^2}{n_{R,0}^3\{e\langle v\sigma_p\rangle p_{R,0}\}^{3/2}\{\epsilon\mu_p N_R\}^{1/2}}=\mathcal{E}^*x^*. \tag{33}$$

In terms of these variables, the dimensionless current-voltage characteristic is given by a plot of $1/w_c$ versus v_c/w_c^2:

$$\frac{1}{w_c}=\frac{aN_R}{e\langle v\sigma_p\rangle n_{R,0}^2 p_{R,0}L}J\,,$$
$$\frac{v_c}{w_c^2}=\frac{1}{n_{R,0}L^2}\left\{\frac{\epsilon\mu_p N_R}{ep_{R,0}\langle v\sigma_p\rangle}\right\}^{1/2}V. \tag{34}$$

The separate regions are now characterized as follows.

Region I. $(0\leq w\leq w_1)$:

Equation (10), characterizing this region, becomes

$$\frac{udu}{u+C}=Ddw;\quad C=\left\{\frac{\epsilon\langle v\sigma_p\rangle p_{R,0}}{ea^2\mu_p N_R}\right\}^{1/2},$$
$$D=\langle v\sigma_n\rangle N_R\left\{\frac{a^2\epsilon N_R}{e\langle v\sigma_p\rangle n_{R,0}^2 p_{R,0}\mu_p}\right\}^{1/2}, \tag{35}$$

which has the solution

$$u-C\ln[(u+C)/C]=Dw\,, \tag{36}$$

satisfying the boundary condition (6), namely, $u=0$ at $w=0$.

The right-hand end of region I is at w_1 corresponding to x_1 given by (14), namely,

$$w_1=C/D\{a-\ln(1+a)\}\,,\quad u_1=aC\,, \tag{37}$$

where (10) has also been used.

The dimensionless voltage V_{I} across region I is, from (15),

$$v_{\mathrm{I}}=(C^2/D)\{\tfrac{1}{2}a^2-a+\ln(1+a)\}\,. \tag{38}$$

For $J>J_M$, $V>V_M$ with J_M, V_M given by (16), there is only the region I in the insulator terminating at $w_c<w_1$, where $w_c=w(L)$. The corresponding dimensionless forms of (18) and (19) are, respectively,

$$w_c=(C/D)\{a(u_c/u_1)-\ln[a(u_c/u_1)+1]\} \tag{39}$$

and

$$v_c=(C^2/D)\{\tfrac{1}{2}a^2(u_c/u_1)^2-a(u_c/u_1)+\ln[a(u_c/u_1)+1]\}\,, \tag{40}$$

with u_1 given by (37). Equations (39) and (40) are used in (34) to obtain the dimensionless $I-V$ characteristic.

At large injection levels, $u_c\ll u_1$, $w_c\simeq u_c^2/2CD$ and $v_c\simeq u_c^3/3CD$. The corresponding $I-V$ characteristic is

$$u_c\ll u_1:\quad (v_c/w_c^2)^2\simeq(8/9)CD(1/w_c)\,, \tag{41}$$

which is the dimensionless form of (20).

Region II. $(w_1\leq w\leq w_2)$:

Equation (26), characterizing this region, becomes

$$du/(u+E)=dw\,,\quad E=\left\{\frac{\epsilon\langle v\sigma_p\rangle n_{R,0}^2}{e\mu_p p_{R,0}N_R}\right\}^{1/2}, \tag{42}$$

with solution

$$w-w_1=\ln\frac{u+E}{u_1+E}. \tag{43}$$

The right-hand end of region II is specified by (27) which, using (42), can be written

$$u_2(u_2+E)=1\,,\quad \text{with}\quad u_2=u(w_2)\,. \tag{44}$$

Using (44) in (43) we get

$$w_2-w_1=\ln\frac{u_2+E}{u_1+E}=-\ln u_2(u_1+E)\,. \tag{45}$$

For the voltage v_{II} across region II we have, using (42) and (45),

$$v_{\mathrm{II}}=\int_{w_1}^{w_2}udw=\int_{u_1}^{u_2}u\frac{dw}{du}du$$
$$=u_2-u_1+E\ln\{u_2(u_1+E)\}\,. \tag{46}$$

Region III. ($w_2 \le w \le w_c$):

Equation (31), characterizing this region, becomes

$$u(d^2u/dw^2) = -1. \quad (47)$$

It is most convenient to write the solution to this equation in a parametric form, namely,

$$u = u_m \exp(-s^2), \quad w - w_2 = \sqrt{2}u_m \int_{s_2}^{s} ds \exp(-s^2). \quad (48)$$

In verifying that (48) is a solution to (47), note that $du/ds = -2\ su$, $dw/ds = \sqrt{2}u$, $du/dw = (du/ds)/(dw/ds) = -\sqrt{2}s$, and finally,

$$(d/dw)(du/dw) = -\sqrt{2}ds/dw = -1/u.$$

At the left-hand end of region III, (48) gives

$$u_2 = u_m \exp(-s_2^2), \quad (49)$$

where u_2 is given by (44) because of our requirement of continuity of u at w_2.

Since u increases monotonically with w in regions I and II, from (36) and (43), it follows that u must reach its maximum, $u = u_m$ at $s = 0$, inside region III. Hence s_2 must be negative: $s_2 = -|s_2|$. Further the boundary condition $\mathcal{E} = 0$ at $x = L$, that is, $u = 0$ at $w = w_c$ corresponds to $s_c = s(w_c) = \infty$ in (48). Thus, at $w = w_c$, (48) can be written as

$$w_c - w_2 = \sqrt{2}u_2 \exp(s_2^2) \int_{-|s_2|}^{\infty} ds \exp(-s^2). \quad (50)$$

For the voltage v_{III} across region III, we have

$$v_{\text{III}} = \int_{w_2}^{w_c} u dw = \int_{-|s_2|} dsu(dw/ds)$$

$$= u_2^2 \exp(2s_2^2) \int_{-\sqrt{2}|s_2|}^{\infty} ds \exp(-s^2), \quad (51)$$

where we have used (48) and the identity

$$\sqrt{2} \int_{-|s_2|}^{\infty} ds \exp(-2s^2) = \int_{-\sqrt{2}|s_2|}^{\infty} ds \exp(-s^2).$$

At sufficiently low currents, regions I and II occupy a negligible fraction of the insulator. Therefore, in the limit of vanishing current we can extend region III right up to the anode, that is, take $w_2 = 0$, $|s_2| = \infty$ and $v_{\text{III}} = v_c$ in (50) and (51). This gives the threshold voltage

$$\left(\frac{v_c}{w_c^2}\right)_{\text{TH}} = \left[2\int_{-\infty}^{\infty} ds \exp(-s^2)\right]^{-1} = \frac{1}{(4\pi)^{1/2}},$$

$$V_{\text{TH}} = \left\{\frac{e\langle v\sigma_p\rangle L^4 n_{R,0}^2 p_{R,0}}{4\pi\epsilon\mu_p N_R}\right\}^{1/2}. \quad (52)$$

Ashley[3] has an analytic result for V_{TH} valid under somewhat more general conditions:

$$V_{\text{TH}} = F\frac{L^2}{2\mu_p\tau_{p,1}}, \quad \tau_{p,1} = \frac{1}{\langle v\sigma_p\rangle n_{R,0}},$$

$$F = \frac{\Gamma(2\alpha+1)\Gamma(2\beta+1)/\Gamma(2\alpha+2\beta+1)}{[\Gamma(\alpha+1)\Gamma(\beta+1)/\Gamma(\alpha+\beta+1)]^2},$$

$$\alpha = \frac{\tau_h}{t_{n,\Omega}}, \quad \beta = \frac{\tau_{p,1}}{t_{p,\Omega}}\frac{n_{R,0}}{N_R}, \quad (53)$$

$$t_{n,\Omega} = \frac{\epsilon}{en_{R,0}\mu_n}, \quad t_{p,\Omega} = \frac{\epsilon}{ep_{R,0}\mu_p},$$

where $\Gamma(x)$ is the well-known γ function. For $\alpha \gg 1$, $\beta \gg 1$ the use of Stirling's approximation gives

$$\alpha \gg 1, \quad \beta \gg 1: \quad F \simeq [(\beta/\pi)(1+\beta/\alpha)]^{1/2}. \quad (54)$$

For $\langle v\sigma_p\rangle \gg \langle v\sigma_n\rangle$, $\tau_h \gg n_{R,0}\tau_{p,1}/N_R$; and for $n_{R,0} \gtrsim \frac{1}{2}N_R$ and $\mu_p \lesssim \mu_n$, $t_{n,\Omega} < t_{p,\Omega}$ so that $\alpha \gg \beta$. Further, if $p_{R,0}$ is large enough that $\beta \gg 1$, then $F \simeq (\beta/\pi)^{1/2}$ which, inserted into (54), gives (52). Thus Ashley's more general threshold result (53) reduces precisely to our very simple derived result (52) in the appropriate limit.

We return to the finite-current problem. Writing $w_c = (w_c - w_2) + (w_2 - w_1) + w_1$ and using (37), (45), and (50), we obtain

$$w_c = \sqrt{2}u_2 \exp(s_2^2) \int_{-|s_2|}^{\infty} ds \exp(-s^2) - \ln u_2(aC+E)$$

$$+ (C/D)\{a - \ln(1+a)\}, \quad (55)$$

where, from (44)

$$u_2 = \tfrac{1}{2}E[-1 + (1+4/E^2)^{1/2}]. \quad (56)$$

Since $v_c = v_{\text{I}} + v_{\text{II}} + v_{\text{III}}$ we have, from (38), (46), and (51),

$$v_c = (C^2/D)\{\tfrac{1}{2}a^2 - a + \ln(1+a)\}$$

$$+ (u_2 - aC) + E \ln u_2(aC+E)$$

$$+ u_2^2 \exp(2s_2^2) \int_{-\sqrt{2}|s_2|}^{\infty} ds \exp(-s^2). \quad (57)$$

Equations (55) and (57), used in (34), gives a parametric representation for the I–V characteristic with $|s_2|$ as the variable parameter. Further detailed results require numerical computation. The I–V characteristic for one choice of parameters is shown in Fig. 3; the corresponding parameter values are $\epsilon/\epsilon_0 = 12$, $\mu_n = \mu_p = 10^4$ cm^2/V sec, $n_{R,0} = p_{R,0} = 5\times10^{14}$ cm^{-3}, $\langle v\sigma_n\rangle = 10^{-9}$, and $\langle v\sigma_p\rangle = 10^{-7}$ cm^3/sec, corresponding to $a = 1$, $C = E = 5.8\times10^{-3}$, and $0 = 2.3\times10^{-4}$. The threshold voltage is $(v_c/w_c^2)_{\text{TH}} = 0.28$ and the turn-around voltage is $(v_c/w_c^2)_M = 2.1\times10^{-5}$, the turn-around current being

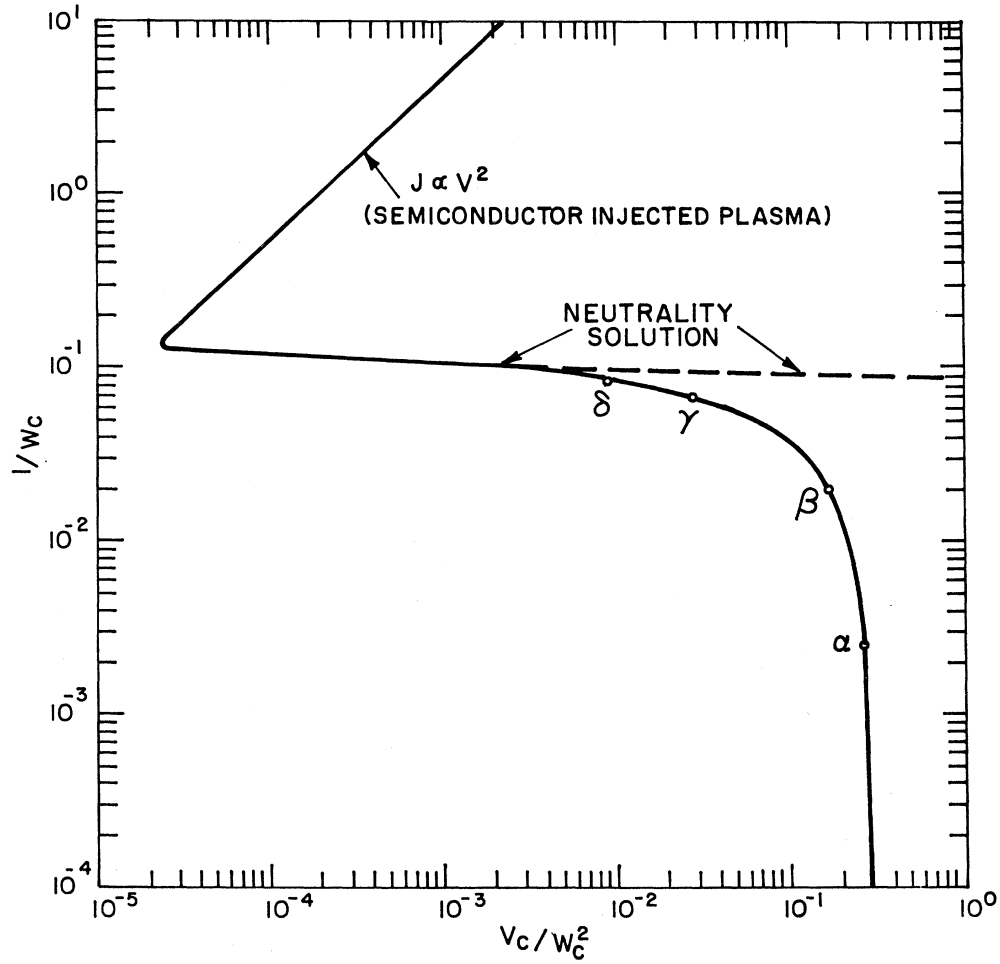

FIG. 3. Prototype universal current-voltage characteristic (solid line) for double injection into an insulator with a single set of recombination centers partially filled in thermal equilibrium. $J \propto 1/w_c$ and $V \propto v_c/w_c^2$.

$(1/w_c)_M = 0.13$. At the point labeled α, region I occupies almost 2% of the insulator, region II 1%, and region III 97%; essentially 100% of the applied voltage is across region III. At point β, regions I and II occupy 14 and 7%, respectively, of the insulator but still absorb negligible voltage; region III occupies 79% of the insulator and absorbs 100% of the voltage. At point γ, region I occupies 48% of the insulator and absorbs 4% of the voltage, region II occupies 26% of the insulator and absorbs 14% of the voltage, and region III occupies 26% of the insulator and absorbs 82% of the voltage. At point δ, region I occupies 56% of the insulator and absorbs 8% of the voltage, region II occupies 29% of the insulator and absorbs 33% of the voltage, and region III occupies 15% of the insulator and absorbs 59% of the voltage. The dashed curve in Fig. 3 is the neutrality-determined solution (regions I and II assumed to fill the insulator), which, extrapolated to zero current yields an infinite threshold voltage.[2]

The physical origin of the negative resistance is easily understood and was discussed in I. At low current levels, the very short hole lifetime $\tau_{p,l}$ constitutes a "recombination barrier" to penetration of the insulator by the injected holes. As the current increases, more and more of the insulator, working out from the anode, is converted, by hole capture, into a region of long hole lifetime τ_h. Penetration of this region by holes is possible at lower voltage drop, hence the decreasing voltage with increasing current. In terms of the regional-approximation analysis, with increasing current the low-voltage region I grows at the expense of the higher-voltage regions II and III.

Because of the empty recombination centers there will also be, in this problem, a threshold voltage for one-carrier electron space-charge-limited current flow, namely, the trap-filled-limit voltage[12] $V_{\mathrm{TFL}} = ep_{R,0}L^2/2\epsilon = L^2/2\mu_p t_{p,\Omega}$, in the notation of (53). For $\alpha \gg \beta \gg 1$ and $n_{R,0} > N_R/2$, $V_{\mathrm{TH}} \simeq (1/\pi\beta)^{1/2} V_{\mathrm{TFL}} < V_{\mathrm{TFL}}$. In this case there will be no electron space-charge current preceding the double-injection current. Where $V_{\mathrm{TFL}} < V_{\mathrm{TH}}$ the trap-filled limit,[12] electron space-charge-limited I–V characteristic will be observed prior to the onset of double injection. [This initial regime in the current-voltage characteristic did not emerge from our analysis, nor from Ashley's,[3] because of our neglect of thermal reemission of electrons from the recombination centers back into the conduction band, assumption (iv). That is, with empty centers $p_{R,0}$ present initially, and the thermal reemission of electrons neglected, electron injection is suppressed until holes are also injected.] Since the trap-filled limit electron flow prior to double injection completely fills the recombination centers, we would expect the double-injection regime in the I–V characteristic to be described by the analysis of Sec. III.

III. RECOMBINATION CENTERS COMPLETELY FILLED, INITIALLY

This is the problem originally studied in I (under the assumption of local neutrality). There is a single set of recombination centers lying sufficiently below the Fermi level that, effectively, they are completely occupied in thermal equilibrium. The energy-band diagram of Fig. 1 illustrates the situation if we relocate the level E_t (or F_0) such that $F_0-E_t \gg kT$. As in the previous problem, we again assume $\bar{\sigma}_p \gg \bar{\sigma}_n$.

Equations (1)–(6) describe equally well the present problem if, in the Poisson equation, we now take $p_{R,0}=0$, that is, if we write

$$(\epsilon/e)(d\mathcal{E}/dx)=p+p_R-n. \qquad (2')$$

Hereafter, when we refer to the defining Eqs. (1)–(6) it is understood that (2′) replaces (2). The same assumptions (i)–(vi) made in the previous problem are again made here.

The analytic tractability of this problem, demonstrated in I with the additional assumption of local neutrality, no longer obtains when this assumption is dropped. We therefore resort once again to the regional-approximation method to obtain an approximate analytic solution. At low currents this is a four-region problem as illustrated schematically in Fig. 4. Entering at the anode, at $x=0$, local neutrality obtains in the first two regions, I and II, and indeed these two regions are very similar to their counterparts in the previous problem. In region I, the recombination centers have been largely depopulated by hole capture: $p_R \simeq N_R$ and $n \gtrsim N_R$. Region I ends at $x=x_1$, where $n(x_1)=N_R$. In region II, the electron transfer from the recombination centers to the conduction band is incomplete: $N_R > n \gg p$. Region II ends where the assumption of local neutrality runs out of self-consistency, that is, at $x=x_2$, where $n(x_2)=(\epsilon/e)(d\mathcal{E}/dx)_2$. Regions III and IV, encompassing the remaining volume of the insulator, are dominated by space charge. In region IV, adjacent to the cathode at $x=L$, the recombination centers have their thermal occupancy, that is, they are completely filled, and so the negative space in this region is necessarily in the conduction band, hence free to carry current. An immediate conclusion is that at arbitrarily low voltage a trap-free one-carrier (electron) space-charge-limited current can flow through the insulator. This is in marked contrast to the previous problem, where the trapping of space charge at low current levels led to a voltage threshold for current flow. In this respect, our use of assumption (iv) in this problem has less drastic consequences than in the previous problem. Region III is a transition region needed to match the local-neutrality-dominated region II to the free-electron space-charge-dominated region IV.

[12] M. A. Lampert, Phys. Rev. **103**, 1648 (1956).

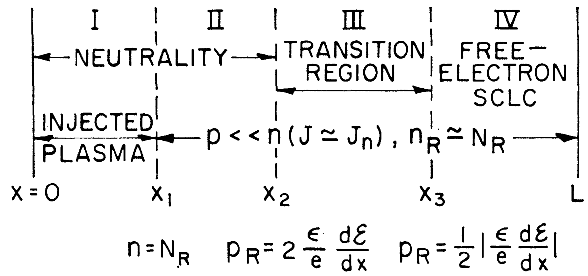

FIG. 4. Schematic regional-approximation diagram for the problem of double injection into an insulator with a single set of recombination centers completely filled in thermal equilibrium.

We proceed to a detailed consideration of the four regions.

Region I. $(0 \le x \le x_1): n > N_R,\ p_R \simeq N_R$.

This region is essentially identical to region I of the previous problem. The Eqs. (7)–(12) hold if only we replace $n_{R,0}$ by N_R, that is, take $p_{R,0}=0$. At the right-hand end of region I we have, in place of (13)–(15),

$$x=x_1:\quad n_1=n(x_1)=N_R, \qquad (58)$$

$$x=(S/T)\{a-\ln(1+a)\},\quad S=J/e\mu_p N_R \qquad (59)$$

$$\mathcal{E}_1=aS,\quad V_{\rm I}=(S^2/T)\{\tfrac{1}{2}a^2-a+\ln(a+1)\}, \qquad (60)$$

with T given by (10). Note that S replaces S_0 in (10)–(12).

The current and voltage at which region I just fills the insulator is again denoted by J_M, V_M and are given by

$$J_M=h(a)(eN_RL/\tau_h),\quad V_M=[1/g(a)](L^2/\mu_p\tau_h), \qquad (61)$$

with τ_h given by (9), $h(a)$ and $g(a)$ by (16). Note, from (16) and (61), that V_M is the same for both problems. Once again, J_M and V_M mark the high-current low-voltage end of the negative-resistance regime. Remarkably, the J_M, V_M given by (61) are exactly the same as obtained from the rigorous neutrality theory of I [Eqs. (B9) and (B10)].

With J_M, V_M now given by (61), (17) holds precisely as written. Equations (18) and (19) hold with S replacing S_0: (20) holds with N_R replacing $n_{R,0}$.

Region II. $(x_1 \leq x \leq x_2): N_R > n \gg p$.

This region is the counterpart of region II of the previous problem. The characteristic differential equation for this region is obtained directly from (25) by taking $p_{R,0}=0$, $n_{R,0}=N_R$:

$$-\frac{dn}{n^2} \simeq \frac{eN_R\langle v\sigma_p\rangle}{aJ}dx. \tag{62}$$

The solution satisfying the boundary condition: $n=N_R$ at $x=x_1$ (continuity of n across the plane x_1), is

$$\frac{1}{n} = \frac{1}{N_R} + \frac{eN_R\langle v\sigma_p\rangle}{aJ}(x-x_1). \tag{63}$$

The field intensity $\mathcal{E}$ is given by

$$\mathcal{E} \simeq \frac{J}{e\mu_n}\frac{1}{n} = \frac{J}{e\mu_n N_R} + \frac{N_R\langle v\sigma_p\rangle}{\mu_p}(x-x_1). \tag{64}$$

Just as with the previous problem, the right-hand terminus x_2 of region II is taken where the neglected space charge overtakes the retained terms in the Poisson equation:

$$x=x_2:\quad n_2=n(x_2)=(\epsilon/e)(d\mathcal{E}/dx)_2=p_{R,2}=p_R(x_2). \tag{65}$$

For the voltage drop V_{II} across region II we have

$$v_{\text{II}} = \int_{x_1}^{x_2} \mathcal{E}dx = \frac{J}{e\mu_n N_R}(x_2-x_1) + \frac{N_R\langle v\sigma_p\rangle}{2\mu_p}(x_2-x_1)^2. \tag{66}$$

Taking $x_2=L$ in (65), this defines a critical current and voltage, J_N and V_N, given by the following:

$$J_N = \frac{\epsilon N_R^2\langle v\sigma_p\rangle^2 L}{2a\mu_p\{\frac{1}{2}-aA+(A/B)[a-\ln(1+a)]\}},\quad A=\frac{\epsilon\langle v\sigma_p\rangle}{2ea\mu_p},\quad B=a\frac{\langle v\sigma_n\rangle}{\langle v\sigma_p\rangle}$$
$$V_N = \frac{N_R\langle v\sigma_p\rangle L^2\{\frac{1}{2}(\frac{1}{4}-a^2A^2)+(A^2/B)[\frac{1}{2}a^2-a+\ln(1+a)]\}}{\mu_p\{\frac{1}{2}-aA+(A/B)[a-\ln(1+a)]\}^2}, \tag{67}$$

where $V_N = V_{\text{I}} + V_{\text{II}}(x_2=L)$.

For $J_N < J < J_M$ (or $V_N > V > V_M$), $x_1 < L < x_2$ and x_2 must be replaced by L in (66). Combining (66), as changed, with (60) we obtain for the I–V characteristic

$$J_N \leq J \leq J_M:\quad V = V_M\{(J/J_M)^2 + f(a)(J/J_M)(1-J/J_M)\} + V_{\text{TH}}(1-J/J_M)^2,$$
$$f(a) = \frac{a\{a-\ln(1+a)\}}{\frac{1}{2}a^2-a+\ln(1+a)},\quad V_{\text{TH}} = \frac{\langle v\sigma_p\rangle L^2 N_R}{2\mu_p} = \frac{L^2}{2\mu_p\bar{\tau}_{p,1}},\quad \bar{\tau}_{p,1} = \frac{1}{\langle v\sigma_p\rangle N_R}. \tag{68}$$

Note that $f(a)$ is almost a constant, of order unity; it varies from $\frac{3}{2}$ at $a \ll 1$ to 2 at $a \gg 1$. Comparing the coefficients V_{TH} and V_M in (68), we have, from (61) and (16), $V_{\text{TH}}/V_M = (\frac{1}{2}g(a))\langle v\sigma_p\rangle/\langle v\sigma_n\rangle$. From (16), $g(a)$ decreases monotonically from $\frac{3}{8}$ at $a \ll 1$ to $1/a$ at $a \gg 1$. Thus, unless a is very large, $V_{\text{TH}} \gg V_M$. In this case, (68) can be further simplified to

$$J_N \leq J \leq J_M \quad\text{and}\quad V_{\text{TH}} \gg V_M:$$
$$V \simeq V_M(J/J_M)^2 + V_{\text{TH}}(1-J/J_M)^2. \tag{69}$$

The current-controlled negative resistance is clearly exhibited by (68) and (69), namely, voltage decreasing monotonically as J increases from J_N to J_M. Equations (68) and (69), practically speaking, have the identical physical content as the rigorous neutrality solution of I, namely, (A14) and (A15), and are very much simpler.

If the neutrality solution is estrapolated, beyond its domain of validity, down to zero current, it gives a threshold voltage for current flow, namely, V_{TH}, hence the subscript "TH." In actuality, for $J < J_N$, $x_2 < L$ and we must take space charge into account by way of including further regions in the problem.

Region III. $(x_2 \leq x \leq x_3): N_R \gg p_R \approx n \gg p$.

The current equation in this region, p being negligible, is given by (22). Space charge is taken into account via the Poisson equation (2′) which here becomes

$$(\epsilon/e)(d\mathcal{E}/dx) = p_R - n. \tag{70}$$

The main role of the space charge in this region is to "turn the field around." The field increases monotonically in regions I and II and decreases monotonically in region IV; hence it must have a maximum in region III. Despite this important role of the space charge $e(p_R-n)$, it is nevertheless true that $p_R \approx n$ throughout region III: $p_{R,2}=n_2$ at the left-hand end of region III and $p_{R,3}=\frac{1}{2}n_3$ at the right-hand end of region III (see the discussion of region IV). Thus, so long as we are not dealing with the difference $n-p_R$, it is legitimate to replace n by p_R in studying region III. Since the derivation of (62) nowhere involved the difference p_R-n it is legitimate to replace n by p_R in this equation:

$$-\frac{dp_R}{p_R^2} = \frac{eN_R\langle v\sigma_p\rangle}{aJ}dx. \tag{71}$$

Equation (71) has the solution

$$\frac{1}{p_R(x)}-\frac{1}{p_{R,2}}=\frac{eN_R\langle v\sigma_p\rangle}{aJ}(x-x_2). \qquad (72)$$

The spatial dependence of $\mathcal{E}$ in region III is given by the solution to the differential equation (70) here rewritten as $(\epsilon/e)(d\mathcal{E}/dx)+J/e\mu_n\mathcal{E}-p_R=0$, with p_R given by (72). Actually, we need not solve this differential equation since our main concern about region III, in this paper, is the voltage drop across it,

$$V_{\text{III}}=\int_{x_2}^{x_3}\mathcal{E}dx.$$

In order to obtain V_{III} it suffices to use a simple interpolation scheme to approximate $\mathcal{E}(x)$, namely, a quadratic (parabolic) expression, as discussed below.

Region IV. $(x_3\leq x\leq L)$: $n\gg p_R\gg p$.

Only the injected free electrons are of significance in this region, so that the Poisson equation (2′) simplifies to

$$(\epsilon/e)(d\mathcal{E}/dx)=-n. \qquad (73)$$

The current equation is again given by (22). The two equations (22) and (73) describe one-carrier (electron) trap-free space-charge-limited current injection. It is easy to check that their solution, corresponding to injection at the cathode at $x=L$, is

$$\mathcal{E}(x)=\left(\frac{2J}{\epsilon\mu_n}\right)^{1/2}(L-x)^{1/2}$$

and (74)

$$n(x)=\left(\frac{\epsilon J}{2e^2\mu_n}\right)\frac{1}{(L-x)^{1/2}}.$$

As the distance from the cathode increases, $n(x)$ decreases as specified by (74). Finally, n gets small enough that it is comparable to the (neglected) trapped-hole density p_R. Thus, a reasonable criterion to fix the left-hand end of region IV is

$x=x_3$:

$$p_{R,3}=p_R(x_3)=\tfrac{1}{2}n_3'=\tfrac{1}{2}n(x_3)=-(\epsilon/2e)(d\mathcal{E}/dx)_3. \qquad (75)$$

For further mathematical discussion it is convenient to switch to dimensionless variables:

$$x=x^+w,\quad \mathcal{E}=\mathcal{E}^+u,\quad V=V^+v,$$

$$x^+=\frac{2a\mu_pJ}{\epsilon N_R^2\langle v\sigma_p\rangle^2},\quad \mathcal{E}^+=\frac{2aJ}{\epsilon N_R\langle v\sigma_p\rangle}, \qquad (76)$$

$$V^+=\frac{4a^2\mu_pJ^2}{\epsilon^2N_R^3\langle v\sigma_p\rangle^3}=\mathcal{E}^+x^+.$$

In terms of these variables, the dimensionless current-voltage characteristic is given by a plot of $1/w_c$ versus v_c/w_c^2:

$$\frac{1}{w_c}=\frac{2a\mu_p}{\epsilon N_R^2\langle v\sigma_p\rangle^2L}J,\quad \frac{v_c}{w_c^2}=\frac{\mu_p}{N_R\langle v\sigma_p\rangle L^2}V. \qquad (77)$$

The separate regions are now characterized as follows.

Region I. $(0\leq w\leq w_1)$:

Equation (10), characterizing this region, and its solution (11) [with S, given by (59), replacing S_0, given by (10)], becomes

$$\frac{udu}{u+A}=Bdw,\quad u-A\ln\frac{u+A}{A}=Bw, \qquad (78)$$

satisfying $u=0$ at $w=0$. A and B are given in (67).

The right-hand end of region I is characterized by (58)–(60). For w_1 corresponding to x_1, and u_1 to $\mathcal{E}_1$, we have

$$w_1=(A/B)\{a-\ln(1+a)\},\quad u_1=aA. \qquad (79)$$

The dimensionless voltage v_{I} across region I is, from (60),

$$v_{\text{I}}=(A^2/B)\{\tfrac{1}{2}a^2-a+\ln(a+1)\}. \qquad (80)$$

Region II. $(w_1\leq w\leq w_2)$:

Equation (62), characterizing this region, and its solution (63), become

$$du/dw=1,\quad u-u_1=w-w_1, \qquad (81)$$

with u_1 and w_1 given by (79).

The right-hand end of region II is characterized by (65). For w_2 corresponding to X_2, and u_2 to $\mathcal{E}_2$, we have

$$w_2=\tfrac{1}{2}-aA+(A/B)\{a-\ln(1+a)\},\quad u_2=\tfrac{1}{2}. \qquad (82)$$

The dimensionless voltage

$$v_{\text{II}}=\int_{w_1}^{w_2}udw$$

across region II is, from (81),

$$v_{\text{II}}=\tfrac{1}{2}(\tfrac{1}{4}-a^2A^2). \qquad (83)$$

Region III. $(w_2\leq w\leq w_3)$:

As previously noted we shall obtain the voltage drop v_{III} across region III through approximation of the field u by an interpolation scheme. This scheme naturally devolves around the quantities u_2, w_2 and u_3, w_3. The former are given in (82). Using (75) in (72) we obtain the important relation

$$-\frac{1}{\frac{1}{2}(du/dw)_3}-\frac{1}{(du/dw)_2}=2(w_3-w_2)$$

or (84)

$$4(w_c-w_3)^{1/2}=1+2(w_3-w_2).$$

Any number of interpolation schemes for u are possible and useful. We have found convenient the following scheme. Let $u=U_2(w)$ be the equation for the tangent to the curve $u=u(w)$ in region II at $w=w_2$; correspondingly, $u=U_3(w)$ is the tangent to the curve $u=u(w)$ in region IV at $w=w_3$. Then

$$U_2(w)=u_2+(w-w_2)(du/dw)_2=u_2+(w-w_2),$$

$$U_3(w)=u_3+(w-w_3)\left(\frac{du}{dw}\right)_3=u_3-\frac{w-w_3}{2(w_c-w_3)^{1/2}}, \quad (85)$$

where we have used (81) and (90).

Further, let $\alpha(w)$ and $\beta(w)$ be two linear weighting functions:

$$\alpha(w)=\frac{w_3-w}{w_3-w_2}, \quad \beta(w)=\frac{w-w_2}{w_3-w_2}, \quad (86)$$

where $\alpha+\beta=1$, $\alpha(w_3)=0$ and $\alpha(w_2)=1$, $\beta(w_3)=1$ and $\beta(w_2)=0$.

Then our interpolation approximation is

$$u(w)=U_2(w)\alpha(w)+U_3(w)\beta(w) \quad (87)$$

or, substituting from (85) and (86),

$$u(w)=(u_2+w-w_2)\frac{(w_3-w)}{(w_3-w_2)} + \left(u_3-\frac{w-w_c}{2(w_c-w_3)^{1/2}}\right)\frac{(w-w_2)}{(w_3-w_2)}. \quad (88)$$

Note that $u(w_2)=u_2$ and $u(w_3)=u_3$, assuring continuity of the $\mathcal{E}$ field across the connecting planes $x=x_2$ and $x=x_3$.

The voltage drop v_{III}, using (88), is

$$v_{\text{III}}=\int_{w_2}^{w_3} u\,dw=\tfrac{1}{2}(u_2+u_3)(w_3-w_2) + \frac{1}{6}\left(1+\frac{1}{2(w_c-w_3)^{1/2}}\right)(w_3-w_2)^2. \quad (89)$$

Region IV. ($w_3 \leq w \leq w_c$):

Equations (73) and (22), characterizing this region, and their solution (74), become

$$\frac{du^2}{dw}=-1, \quad u=(w_c-w)^{1/2}, \qquad \frac{du}{dw}=-\frac{1}{2(w_c-w)^{1/2}}, \quad (90)$$

satisfying $u_c=u(w_c)=0$, that is $\mathcal{E}(L)=0$.

The voltage across region IV is

$$v_{\text{IV}}=\int_{w_3}^{w_c} u\,dw=\tfrac{2}{3}(w_c-w_3)^{3/2}. \quad (91)$$

We are ready now to collect all of the results. The I–V characteristic is given by a plot of $1/w_c$ versus v_c/w_c^2. Writing $w_c=w_2+(w_3-w_2)+(w_c-w_3)$, we have from (82) and (84)

$$\frac{1}{w_c}=\frac{1}{-1-aA+(A/B)\{a-\ln(1+a)\}+(y+1)^2}, \quad (92)$$

$$y=(w_c-w_3)^{1/2}.$$

From (91), $v_{\text{IV}}=2y^3/3$. From (89), $v_{\text{III}}=5y^2/3+\frac{1}{4}y-\frac{1}{4}+1/48y$. Inserting these into the voltage relation: $v_c=v_{\text{I}}+v_{\text{II}}+v_{\text{III}}+v_{\text{IV}}$ we obtain finally, from (83) and (80),

$$\frac{v_c}{w_c^2}=\frac{\frac{1}{2}(\frac{1}{4}-a^2A^2)+(A^2/B)\{\frac{1}{2}a^2-a+\ln(a+1)\}+\frac{2}{3}y^3+\frac{5}{3}y^2+\frac{1}{4}y-\frac{1}{4}+1/48y}{[-1-aA+(A/B)\{a-\ln(a+1)\}+(y+1)^2]^2}. \quad (93)$$

The relations (92) and (93) give a parametric representation of the I–V characteristic in terms of the single parameter y and the material constants a, A, and B, defined in (67).

In terms of the dimensionless variables the critical point J_M, V_M on the I–V characteristic becomes

$$\left(\frac{1}{w_c}\right)_M=\frac{h(a)B}{aA}, \quad \left(\frac{v_c}{w_c^2}\right)_M=\frac{B}{ag(a)}. \quad (94)$$

The I–V characteristic obtained with the assumption of local neutrality (only regions I and II in the insulator) is given by (68) or (69), where we further note the useful relations $V/V_M=(v_c/w_c^2)/(v_c/w_c^2)_M$, $J/J_M=(1/w_c)/(1/w_c)_M$, and $V_{\text{TH}}/V_M=ag(a)/2B$. Corresponding to the voltage threshold V_{TH}, $(v_c/w_c^2)_{\text{TH}}=\frac{1}{2}$.

The electron trap-free square law, $J=9\epsilon\mu_nV^2/8L^3$, becomes $1/w_c=(9/4)(v_c/w_c^2)^2$.

The full I–V characteristic, (92) and (93), is compared with the neutrality-based characteristic (68) for two cases in Fig. 5. The solid straight line of slope two in the figure is the electron trap-free square law. The two solid curves are neutrality-based characteristics and the two dashed curves are the corresponding full (space-charge-included) characteristics. The material parameters chosen for the lower pair of curves were $\epsilon/\epsilon_0=12$, $\mu_n=\mu_p=10^4$ cm^2/V sec, $N_R=10^{15}$ cm^{-3}, $\langle v\sigma_n\rangle=9\times10^{-10}$ cm^3/sec, and $\langle v\sigma_p\rangle=3\times10^{-6}$ cm^3/sec,

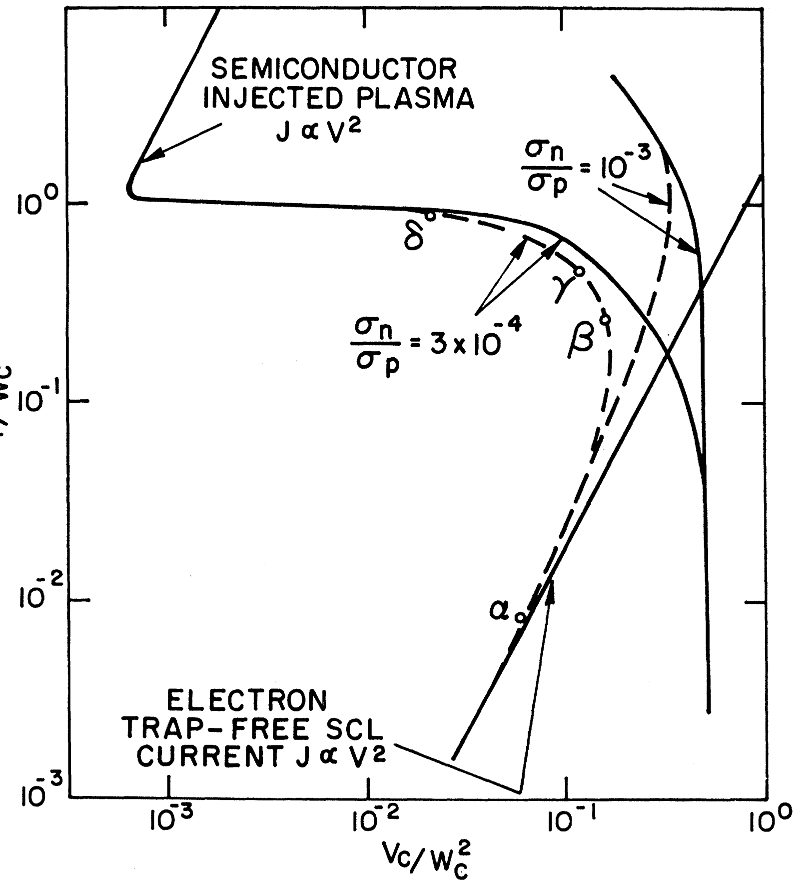

FIG. 5. Prototype universal current-voltage characteristics for double injection into an insulator with a single set of recombination centers completely filled in thermal equilibrium. $J\propto 1/w_c$ and $V\propto v_c/w_c^2$. The two cases calculated are $\sigma_n/\sigma_p=10^{-3}$ and $\sigma_n/\sigma_p=3\times10^{-4}$. The heavy solid lines are the characteristics obtained assuming neutrality; the dashed lines are those obtained including space charge.

as might pertain to a silicon experiment at liquid-nitrogen temperature. The corresponding dimensionless parameters are $a=1$, $A=10^{-3}$, and $B=3\times10^{-4}$. For the upper pair of curves the changes in the material parameters are $\langle v\sigma_n\rangle=10^{-9}$ cm³/sec and $\langle v\sigma_p\rangle=10^{-6}$; correspondingly $A=3\times10^{-4}$ and $B=10^{-3}$.

Some further particulars are of interest. On the lower dashed curve, at point α, the electron space-charge region IV occupies 83% of the crystal and absorbs 81% of the applied voltage. The transition region III takes up the rest of the crystal and the applied voltage, both. Regions I and II are negligible. Going to point β, the neutrality regions I and II are now in the picture. They take 32% of the crystal width, though only 2% of the voltage. Region III takes 43% of the width and 72% of the voltage, region IV, 25% of the width and 26% of the voltage. At point γ, regions I and II now occupy 55% of the width and take 5% of the voltage, region III, 34% of the width and 80% of the voltage, region IV, 11% of the width and 5% of the voltage. At this point, going up on the curve, region IV is shrinking fast. At point δ the two neutrality regions I and II occupy 99% of the crystal width and 98% of the voltage, the residual width and voltage being in the transition region III.

IV. SUMMARY

The problem of double injection with negative resistance due to an injection-level-dependent lifetime has been studied theoretically, including space charge, for two prototype models. Earlier studies, valid at low (or threshold) and high currents have been simplified and extended. A detailed, approximate picture of the evolution of the negative resistance at intermediate currents has been provided through systematic use of the regional-approximation method.